Exercice 1. Calculer les $f'(x)$ lorsque :

1. $f(x)=\mathrm{e}^{2x+1}$
2. $f(x)=x^2\mathrm{e}^{2x+1}$
3. $f(x)=\dfrac{1}{x^2+3x+2}$
4. $f(x)=\dfrac{5x+1}{7x+3}$

Exercice 2. Calculer les limites suivantes :

1. $\lim\limits_{x\to+\infty} x\mathrm{e}^{-x+1}$
2. $\lim\limits_{x\to-\infty} x^2\mathrm{e}^{x}$
3. $\lim\limits_{x\to+\infty} \dfrac{\mathrm{e}^{x}-1}{\mathrm{e}^{2x}-3}$
4. $\lim\limits_{x\to 0} \dfrac{\mathrm{e}^{3x}-1}{x}$

Exercice 3. Déterminer l'équation de la tangente à C_f en x_0 lorsque :

1. $f(x)=\ln(x)$ et $x_0=1$
2. $f(x)=\mathrm{e}^x+1$ et $x_0=0$
3. $f(x)=\sqrt{x+1}$ et $x_0=0$
4. $f(x)=\dfrac{1}{\sqrt{x+1}}$ et $x_0=0$

Exercice 4. Calculer les dérivées de :

1. $\dfrac{x^2+1}{x^2-1}$,
2. $\dfrac{\mathrm{e}^x+1}{\mathrm{e}^{2x}-1}$,
3. $\dfrac{\ln(x)}{\ln(x)+1}$, $\sqrt{x^2+1}$,
4. $\dfrac{1}{\sqrt{x^2+1}}$

Exercice 5. Calculer :

1. $\lim\limits_{x\to 0} \dfrac{\mathrm{e}^{2x}-1}{\mathrm{e}^{x}-1}$,
2. $\lim\limits_{x\to 0} \dfrac{\ln(x+1)}{\ln\left(x^2+1\right)}$,
3. $\lim\limits_{x\to+\infty} \dfrac{\ln(x+1)}{\ln(x)}$
4. $\lim\limits_{x\to+\infty} \ln\left(\mathrm{e}^x+1\right)-x$,
5. $\lim\limits_{x\to+\infty} \dfrac{\sqrt{x^2+1}-x}{\sqrt{x+1}-\sqrt{x}}$

Correction des exercices

Exercice 1.

1. $f'(x)=2\mathrm{e}^{2x+1}\quad \left((\mathrm{e}^u)'=u'\mathrm{e}^u\right)$
2. $f'(x)=2x\mathrm{e}^{2x+1}+2x^2\mathrm{e}^{2x+1}=\left(2x^2+2x\right)\mathrm{e}^{2x+1}$
3. $f'(x)=-\dfrac{2x+3}{x^2+3x+2}\left(\left(\dfrac{1}{u}\right)'=-\dfrac{u'}{u^2}\right)$
4. $f'(x)=\dfrac{5(7x+3)-7(5+1)}{(7x+3)^2}=\dfrac{8}{(7x+3)^2}$

Exercice 2. 1. $\lim_{x\to+\infty} xe^{-x+1} = e\left(xe^{-x}\right)$. Or $\lim_{x\to+\infty} xe^{-x} = 0$, d'où $\lim_{x\to+\infty} xe^{-x+1} = 0$

2. $\lim_{x\to-\infty} x^2\mathrm{e}^x = \lim_{x\to-\infty}\left(x\mathrm{e}^{x/2}\right)^2 = 0$ (croissance comparée : $\lim_{x\to-\infty} x^\alpha \mathrm{e}^{\beta x} = 0$ Si $\beta > 0$)

3.

$$\begin{aligned}\lim_{x\to+\infty}\frac{\mathrm{e}^x-1}{\mathrm{e}^{2x}-3} &= \lim_{x\to+\infty}\frac{\mathrm{e}^x\left(1-\mathrm{e}^{-x}\right)}{\mathrm{e}^{2x}\left(1-3\mathrm{e}^{-2x}\right)}\\ &= \lim_{x\to+\infty}\mathrm{e}^{-x}\frac{1-\mathrm{e}^{-x}}{1-3\mathrm{e}^{-2x}}.\end{aligned}$$

On a $\lim_{x\to+\infty}\mathrm{e}^{-x} = 0$ et $\lim_{x\to+\infty}\frac{1-\mathrm{e}^{-x}}{1-3\mathrm{e}^{-2}x} = 1$ Donc $\lim_{x\to+\infty}\frac{\mathrm{e}^x-1}{\mathrm{e}^{2x}-3} = 0$

4. On a :

$$\begin{aligned}\lim_{x\to0}\frac{\mathrm{e}^{3x}-1}{x} &= \lim_{X\to0}\frac{\mathrm{e}^X-1}{\frac{X}{3}} \text{ où } X = 3x.\\ &= \lim_{X\to0} 3\frac{\mathrm{e}^X-1}{X}\end{aligned}$$

Or $\lim_{X\to0}\frac{\mathrm{e}^X-1}{X} = 1$, d'où $\lim_{x\to0}\frac{\mathrm{e}^{3x}-1}{x} = 3$.

Exercice 3. On note T la tangente à C_f en x_0. $T\ :\ y = f'(x_0)(x-x_0)+f(x_0)$ On a besoin, de $f(x_0)$ et $f'(x_0)$. If faut, donc calculer $f'(x)$ et l'évaluer en x_0.

1. $f'(x) = \frac{1}{x}$, $f(1) = \ln(1) = 0$ et $f'(1) = \frac{1}{1} = 1$. Donc $T\ :\ y = 1(x-1)+0 = x-1$
2. $f'(x) = \mathrm{e}^x$, $f(0) = \mathrm{e}^0+1 = 2$ et $f'(0) = \mathrm{e}^0 = 1$.
   Donc T : $y = 1(x-0)+2 = x+2$
3. $f'(x) = \frac{1}{2\sqrt{x+1}}\cdot\left((\sqrt{u})' = \frac{u'}{2\sqrt{u}}\right)$, $f(0) = \sqrt{0+1} = 1$ et $f'(0) = \frac{1}{2\sqrt{0+1}} = \frac{1}{2}$. Donc $T\ :\ y = \frac{1}{2}(x-0)+1 = \frac{1}{2}x+1$
4. $f'(x) = -\frac{(\sqrt{x+1})'}{\sqrt{x+1}} = -\frac{1}{2\sqrt{x+1}(x+1)} = -\frac{1}{2(x+1)^{3/2}}$,
   $f(0) = \frac{1}{\sqrt{0+1}} = 1$ et $f'(0) = -\frac{1}{2(0+1)^{3/2}} = -\frac{1}{2}$.
   Donc $T\ :\ y = -\frac{1}{2}(x-0)+1 = -\frac{1}{2}x+1$

Exercice 4. 1.

$$\begin{aligned}\left(\frac{x^2+1}{x^2-1}\right)' &= \frac{2x\left(x^2-1\right)-2x\left(x^2+1\right)}{\left(x^2-1\right)^2}\\ &= -\frac{4x}{\left(x^2-1\right)^2}\end{aligned}$$

2.

$$\begin{aligned}\left(\frac{\mathrm{e}^x+1}{\mathrm{e}^{2x}-1}\right)' &= \frac{\mathrm{e}^x\left(\mathrm{e}^{2x}-1\right)-2\left(\mathrm{e}^x+1\right)\mathrm{e}^{2x}}{\left(\mathrm{e}^{2x}-1\right)^2}\\ &= \frac{-e3x-2\mathrm{e}^{2x}-\mathrm{e}^x}{\left(\mathrm{e}^{2x}-1\right)^2}\\ &= -\mathrm{e}^x\frac{e2x+ex+1}{\left(\mathrm{e}^{2x}-1\right)^2} = -\mathrm{e}^x\frac{\left(\mathrm{e}^x+1\right)^2}{\left(\mathrm{e}^{2x}-1\right)^2}\\ &= -\mathrm{e}^x\frac{\left(\mathrm{e}^x+1\right)^2}{(\mathrm{e}^x-1)^2(\mathrm{e}^x+1)^2}\end{aligned}$$

Après simplification obtient
$\left(\frac{e^x+1}{e^{2x}-1}\right)' = -\frac{e^x}{(e^x-1)^2}$.
Autre façon :
On a $\frac{e^x+1}{e^{2x}-1} = u'(v(x))$ où $u(x) = \frac{x+1}{x^2-1}$ et $v(x) = e^x$. Donc $\left(\frac{e^x+1}{e^{2x}-1}\right)' = v'(x)u'(v(x))$.
On a : $u'(x) = \frac{1\left(x^2-1\right)-2x(x+1)}{\left(x^2-1\right)^2} = \frac{-x^2-2x-1}{\left(x^2-1\right)^2}$.
Soit $u'(x) = -\frac{(x+1)^2}{(x-1)^2(x+1)^2} = -\frac{1}{(x-1)^2}$ et $v'(x) = e^x$. Donc $\left(\frac{e^x+1}{e^{2x}-1}\right)' = -\frac{e^x}{\left(e^x-1\right)^2}$

3. On a :

$$\begin{aligned}\left(\frac{\ln(x)}{\ln(x)+1}\right)' &= \frac{\frac{1}{x}(\ln(x)+1)-\frac{1}{x}(\ln(x))}{(\ln(x)+1)^2}\\ &= \frac{1}{x(\ln(x)+1)^2}.\end{aligned}$$

Autre façon :
On a $\frac{\ln(x)}{\ln(x)+1} = u(v(x))$ avec $u(x) = \frac{x}{x+1}$ et $v(x) = \ln(x)$. On a $u'(x) = \frac{1(x+1)-x\times 1}{(x+1)^2} = \frac{1}{(x+1)^2}$ et $v'(x) = \frac{1}{x}$.
Donc

$$\begin{aligned}\left(\frac{\ln(x)}{\ln(x)+1}\right)' &= v'(x)u'(v(x))\\ &= \frac{1}{x}\frac{1}{(\ln(x)+1)^2}\\ &= \frac{1}{x(\ln(x)+1)^2}\end{aligned}$$

4.

$$\begin{aligned}\left(\sqrt{x^2+1}\right)' &= \frac{\left(x^2+1\right)'}{2\sqrt{x^2+1}} \quad \left((\sqrt{u})' = \frac{u'}{2\sqrt{u}}\right)\\ &= \frac{2x}{2\sqrt{x^2+1}}\\ &= \frac{x}{\sqrt{x^2+1}}\end{aligned}$$

$$\begin{aligned}\left(\frac{1}{\sqrt{x^2+1}}\right)' &= -\frac{\left(\sqrt{x^2+1}\right)'}{\left(\sqrt{x^2+1}\right)^2} \left(\frac{1}{u}\right)'\\ &= -\frac{u'}{u^2})\\ &= -\frac{\frac{x}{\sqrt{x^2+1}}}{x^2+1}\\ &= -\frac{x}{\sqrt{x^2+1}\,(x^2+1}\\ &= -\frac{x}{\left(x^2+1\right)^{3/2}}\end{aligned}$$

Autre façon : On a

$$\left(\frac{1}{\sqrt{x^2+1}}\right)' = \left(\left(x^2+1\right)^{-1/2}\right)' = -\frac{1}{2}\times 2x\left(x^2+1\right)^{-1/2-1} = -\frac{x}{\left(x^2+1\right)^{3/2}}.$$

Exercice 5. 1. $\lim_{x\to 0}\frac{e^{2x}-1}{e^x-1} = \lim_{x\to 0}\frac{\left(e^x-1\right)\left(e^x+1\right)}{e^x-1} = \lim_{x\to 0} e^x+1 = 2$

2. $\lim\limits_{x\to 0^+} \dfrac{\ln(x+1)}{\ln(x^2+1)} = \lim\limits_{x\to 0} \dfrac{\ln(x+1)}{x} \times \dfrac{x^2}{\ln(x^2+1)} \times \dfrac{1}{x}$.
Or $\lim\limits_{x\to 0} \dfrac{\ln(x+1)}{x} = 1$ et $\lim\limits_{x\to 0} \dfrac{\ln(x^2+1)}{x^2} = \lim\limits_{X\to 0} \dfrac{\ln(X+1)-1}{X}$ (où $X = x^2$) et $\lim\limits_{x\to 0^+} \dfrac{1}{x} = +\infty$, d'où $\lim\limits_{x\to 0^+} \dfrac{\ln(x+1)}{\ln(x^2+1)} = +\infty$
3.
$$\begin{aligned}\lim_{x\to+\infty} \frac{\ln(x+1)}{\ln(x)} &= \lim_{x\to+\infty} \frac{\ln\left(x\left(1+\frac{1}{x}\right)\right)}{\ln(x)} \\ &= \lim_{x\to+\infty} \frac{\ln(x)+\ln\left(1+\frac{1}{x}\right)}{\ln(x)} \\ &= \lim_{x\to+\infty} \left(1+\frac{\ln\left(1+\frac{1}{x}\right)}{\ln(x)}\right) \\ &= 1\end{aligned}$$
Car $\lim\limits_{x\to+\infty} \ln\left(1+\dfrac{1}{x}\right) = 0$ et $\lim\limits_{x\to+\infty} \ln(x) = +\infty$.
4. On a
$$\begin{aligned}\ln\left(e^x+1\right)-x &= \ln\left(e^x\left(1+e^{-x}\right)\right)-x \\ &= \ln\left(e^x\right)+\ln\left(1+e^{-x}\right)-x \\ &= x+\ln\left(1+e^{-x}\right)-x\end{aligned}$$
Donc $\lim\limits_{x\to+\infty} \ln\left(e^x+1\right)-x = \lim\limits_{x\to+\infty} \ln\left(1+e^{-x}\right) = 0$.
5. On a $\sqrt{x^2+1}-x = \dfrac{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}+x} = \dfrac{1}{\sqrt{x^2+1}+x}$ et $\sqrt{x+1}-\sqrt{x} = \dfrac{(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+x)}{\sqrt{x+1}+\sqrt{x}} = \dfrac{1}{\sqrt{x+1}+\sqrt{x}}$
Donc
$$\begin{aligned}\lim_{x\to+\infty} \frac{\sqrt{x^2+1}-x}{\sqrt{x+1}-\sqrt{x}} &= \lim_{x\to+\infty} \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x^2+1}+x} \\ &= \lim_{x\to+\infty} \frac{\sqrt{x}}{x} \frac{\sqrt{1+\frac{1}{x}}+1}{\sqrt{1+\frac{1}{x^2}+1}+1} = 0\end{aligned}$$
Or $\lim\limits_{x\to+\infty} \dfrac{\sqrt{x}}{x} = \lim\limits_{x\to+\infty} \dfrac{1}{\sqrt{x}} = 0$ et $\lim\limits_{x\to+\infty} \dfrac{\sqrt{1+\frac{1}{x}}+1}{\sqrt{1+\frac{1}{x}}+1} = 2$. Donc $\lim\limits_{x\to+\infty} \dfrac{\sqrt{x^2+1}-x}{\sqrt{x+1}-\sqrt{x}} = 0$.
C' est plus simple en 2ème année avec de nouvelles méthodes.