

Exercice 1. Calculer les $f'(x)$ lorsque :

1. $f(x) = e^{2x+1}$
2. $f(x) = x^2 e^{2x+1}$
3. $f(x) = \frac{1}{x^2 + 3x + 2}$
4. $f(x) = \frac{5x + 1}{7x + 3}$

Exercice 2. Calculer les limites suivantes :

1. $\lim_{x \rightarrow +\infty} x e^{-x+1}$
2. $\lim_{x \rightarrow -\infty} x^2 e^x$
3. $\lim_{x \rightarrow +\infty} \frac{e^x - 1}{e^{2x} - 3}$
4. $\lim_{x \rightarrow 0} \frac{e^{3x} - 1}{x}$

Exercice 3. Déterminer l'équation de la tangente à C_f en x_0 lorsque :

1. $f(x) = \ln(x)$ et $x_0 = 1$
2. $f(x) = e^x + 1$ et $x_0 = 0$
3. $f(x) = \sqrt{x+1}$ et $x_0 = 0$
4. $f(x) = \frac{1}{\sqrt{x+1}}$ et $x_0 = 0$

Exercice 4. Calculer les dérivées de :

1. $\frac{x^2 + 1}{x^2 - 1}$,
2. $\frac{e^x + 1}{e^{2x} - 1}$,
3. $\frac{\ln(x)}{\ln(x) + 1}, \sqrt{x^2 + 1}$,
4. $\frac{1}{\sqrt{x^2 + 1}}$

Exercice 5. Calculer :

1. $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{e^x - 1}$,
2. $\lim_{x \rightarrow 0} \frac{\ln(x+1)}{\ln(x^2+1)}$,
3. $\lim_{x \rightarrow +\infty} \frac{\ln(x+1)}{\ln(x)}$
4. $\lim_{x \rightarrow +\infty} \ln(e^x + 1) - x$,
5. $\lim_{x \rightarrow +\infty} \frac{\sqrt{x^2+1} - x}{\sqrt{x+1} - \sqrt{x}}$

Correction des exercices

- Exercice 1.**
1. $f'(x) = 2e^{2x+1} \quad ((e^u)' = u' e^u)$
 2. $f'(x) = 2x e^{2x+1} + 2x^2 e^{2x+1} = (2x^2 + 2x) e^{2x+1}$
 3. $f'(x) = -\frac{2x+3}{x^2+3x+2} \left(\left(\frac{1}{u} \right)' = -\frac{u'}{u^2} \right)$
 4. $f'(x) = \frac{5(7x+3) - 7(5+1)}{(7x+3)^2} = \frac{8}{(7x+3)^2}$

Exercice 2. 1. $\lim_{x \rightarrow +\infty} x e^{-x+1} = e(xe^{-x})$. Or $\lim_{x \rightarrow +\infty} x e^{-x} = 0$, d'où $\lim_{x \rightarrow +\infty} x e^{-x+1} = 0$

2. $\lim_{x \rightarrow -\infty} x^2 e^x = \lim_{x \rightarrow -\infty} (x e^{x/2})^2 = 0$ (croissance comparée : $\lim_{x \rightarrow -\infty} x^\alpha e^{\beta x} = 0$
Si $\beta > 0$)

3.

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{e^x - 1}{e^{2x} - 3} &= \lim_{x \rightarrow +\infty} \frac{e^x (1 - e^{-x})}{e^{2x} (1 - 3e^{-2x})} \\ &= \lim_{x \rightarrow +\infty} e^{-x} \frac{1 - e^{-x}}{1 - 3e^{-2x}}. \end{aligned}$$

On a $\lim_{x \rightarrow +\infty} e^{-x} = 0$ et $\lim_{x \rightarrow +\infty} \frac{1 - e^{-x}}{1 - 3e^{-2x}} = 1$ Donc $\lim_{x \rightarrow +\infty} \frac{e^x - 1}{e^{2x} - 3} = 0$

4. On a :

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^{3x} - 1}{x} &= \lim_{x \rightarrow 0} \frac{e^X - 1}{\frac{X}{3}} \text{ où } X = 3x. \\ &= \lim_{X \rightarrow 0} 3 \frac{e^X - 1}{X} \end{aligned}$$

Or $\lim_{X \rightarrow 0} \frac{e^X - 1}{X} = 1$, d'où $\lim_{x \rightarrow 0} \frac{e^{3x} - 1}{x} = 3$.

Exercice 3. On note T la tangente à C_f en x_0 . $T : y = f'(x_0)(x - x_0) + f(x_0)$. On a besoin, de $f(x_0)$ et $f'(x_0)$. Il faut, donc calculer $f'(x)$ et l'évaluer en x_0 .

1. $f'(x) = \frac{1}{x}$, $f(1) = \ln(1) = 0$ et $f'(1) = \frac{1}{1} = 1$. Donc $T : y = 1(x - 1) + 0 = x - 1$

2. $f'(x) = e^x$, $f(0) = e^0 + 1 = 2$ et $f'(0) = e^0 = 1$.

Donc $T : y = 1(x - 0) + 2 = x + 2$

3. $f'(x) = \frac{1}{2\sqrt{x+1}} \cdot \left((\sqrt{u})' = \frac{u'}{2\sqrt{u}} \right)$, $f(0) = \sqrt{0+1} = 1$ et $f'(0) = \frac{1}{2\sqrt{0+1}} = \frac{1}{2}$. Donc $T : y =$
 $\frac{1}{2}(x - 0) + 1 = \frac{1}{2}x + 1$

4. $f'(x) = -\frac{(\sqrt{x+1})'}{\sqrt{x+1}} = -\frac{1}{2\sqrt{x+1}(x+1)} = -\frac{1}{2(x+1)^{3/2}}$,

$f(0) = \frac{1}{\sqrt{0+1}} = 1$ et $f'(0) = -\frac{1}{2(0+1)^{3/2}} = -\frac{1}{2}$.

Donc $T : y = -\frac{1}{2}(x - 0) + 1 = -\frac{1}{2}x + 1$

Exercice 4. 1.

$$\begin{aligned} \left(\frac{x^2 + 1}{x^2 - 1} \right)' &= \frac{2x(x^2 - 1) - 2x(x^2 + 1)}{(x^2 - 1)^2} \\ &= -\frac{4x}{(x^2 - 1)^2} \end{aligned}$$

2.

$$\begin{aligned} \left(\frac{e^x + 1}{e^{2x} - 1} \right)' &= \frac{e^x (e^{2x} - 1) - 2(e^x + 1)e^{2x}}{(e^{2x} - 1)^2} \\ &= \frac{-e^{3x} - 2e^{2x} - e^x}{(e^{2x} - 1)^2} \\ &= -e^x \frac{e^{2x} + 2e^x + 1}{(e^{2x} - 1)^2} = -e^x \frac{(e^x + 1)^2}{(e^{2x} - 1)^2} \\ &= -e^x \frac{(e^x + 1)^2}{(e^x - 1)^2 (e^x + 1)^2} \end{aligned}$$

Après simplification obtient

$$\left(\frac{e^x + 1}{e^{2x} - 1}\right)' = -\frac{e^x}{(e^x - 1)^2}.$$

3. On a :

$$\begin{aligned} \left(\frac{\ln(x)}{\ln(x) + 1}\right)' &= \frac{\frac{1}{x}(\ln(x) + 1) - \frac{1}{x}(\ln(x))}{(\ln(x) + 1)^2} \\ &= \frac{1}{x(\ln(x) + 1)^2}. \end{aligned}$$

Autre façon :

On a $\frac{\ln(x)}{\ln(x) + 1} = u(v(x))$ avec $u(x) = \frac{x}{x+1}$ et $v(x) = \ln(x)$. On a $u'(x) = \frac{1(x+1) - x \times 1}{(x+1)^2} = \frac{1}{(x+1)^2}$

et $v'(x) = \frac{1}{x}$.

Donc

$$\begin{aligned} \left(\frac{\ln(x)}{\ln(x) + 1}\right)' &= v'(x)u'(v(x)) \\ &= \frac{1}{x} \frac{1}{(\ln(x) + 1)^2} \\ &= \frac{1}{x(\ln(x) + 1)^2} \end{aligned}$$

4.

$$\begin{aligned} (\sqrt{x^2 + 1})' &= \frac{(x^2 + 1)'}{2\sqrt{x^2 + 1}} \quad \left((\sqrt{u})' = \frac{u'}{2\sqrt{u}}\right) \\ &= \frac{2x}{2\sqrt{x^2 + 1}} \\ &= \frac{x}{\sqrt{x^2 + 1}} \end{aligned}$$

$$\begin{aligned} \left(\frac{1}{\sqrt{x^2 + 1}}\right)' &= -\frac{(\sqrt{x^2 + 1})'}{(\sqrt{x^2 + 1})^2} \left(\frac{1}{u}\right)' \\ &= -\frac{u'}{u^2} \\ &= -\frac{\frac{x}{\sqrt{x^2 + 1}}}{x^2 + 1} \\ &= -\frac{x}{\sqrt{x^2 + 1}(x^2 + 1)} \\ &= -\frac{x}{(x^2 + 1)^{3/2}} \end{aligned}$$

Autre façon : On a

$$\left(\frac{1}{\sqrt{x^2 + 1}}\right)' = \left((x^2 + 1)^{-1/2}\right)' = -\frac{1}{2} \times 2x (x^2 + 1)^{-1/2-1} = -\frac{x}{(x^2 + 1)^{3/2}}.$$

Exercice 5. 1. $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{e^x - 1} = \lim_{x \rightarrow 0} \frac{(e^x - 1)(e^x + 1)}{e^x - 1} = \lim_{x \rightarrow 0} e^x + 1 = 2$

2. $\lim_{x \rightarrow 0^+} \frac{\ln(x+1)}{\ln(x^2+1)} = \lim_{x \rightarrow 0} \frac{\ln(x+1)}{x} \times \frac{x^2}{\ln(x^2+1)} \times \frac{1}{x}$.

Or $\lim_{x \rightarrow 0} \frac{\ln(x+1)}{x} = 1$ et $\lim_{x \rightarrow 0} \frac{\ln(x^2+1)}{x^2} = \lim_{X \rightarrow 0} \frac{\ln(X+1) - 1}{X}$ (où $X = x^2$) et $\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$, d'où

$$\lim_{x \rightarrow 0^+} \frac{\ln(x+1)}{\ln(x^2+1)} = +\infty$$

3.

$$\begin{aligned}
\lim_{x \rightarrow +\infty} \frac{\ln(x+1)}{\ln(x)} &= \lim_{x \rightarrow +\infty} \frac{\ln\left(x\left(1 + \frac{1}{x}\right)\right)}{\ln(x)} \\
&= \lim_{x \rightarrow +\infty} \frac{\ln(x) + \ln\left(1 + \frac{1}{x}\right)}{\ln(x)} \\
&= \lim_{x \rightarrow +\infty} \left(1 + \frac{\ln\left(1 + \frac{1}{x}\right)}{\ln(x)}\right) \\
&= 1
\end{aligned}$$

Car $\lim_{x \rightarrow +\infty} \ln\left(1 + \frac{1}{x}\right) = 0$ et $\lim_{x \rightarrow +\infty} \ln(x) = +\infty$.

4. On a

$$\begin{aligned}
\ln(e^x + 1) - x &= \ln(e^x(1 + e^{-x})) - x \\
&= \ln(e^x) + \ln(1 + e^{-x}) - x \\
&= x + \ln(1 + e^{-x}) - x
\end{aligned}$$

Donc $\lim_{x \rightarrow +\infty} \ln(e^x + 1) - x = \lim_{x \rightarrow +\infty} \ln(1 + e^{-x}) = 0$.

5. On a $\frac{\sqrt{x^2+1}-x}{\sqrt{x}} = \frac{(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{\sqrt{x^2+1}+x}$ et $\sqrt{x+1}-\sqrt{x} = \frac{(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+1)}{\sqrt{x+1}+\sqrt{x}} =$

$$\frac{1}{\sqrt{x+1}+\sqrt{x}}$$

Donc

$$\begin{aligned}
\lim_{x \rightarrow +\infty} \frac{\sqrt{x^2+1}-x}{\sqrt{x+1}-\sqrt{x}} &= \lim_{x \rightarrow +\infty} \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x^2+1}+x} \\
&= \lim_{x \rightarrow +\infty} \frac{\sqrt{x}}{x} \frac{\sqrt{1+\frac{1}{x}}+1}{\sqrt{1+\frac{1}{x^2}}+1+1} = 0
\end{aligned}$$

Or $\lim_{x \rightarrow +\infty} \frac{\sqrt{x}}{x} = \lim_{x \rightarrow +\infty} \frac{1}{\sqrt{x}} = 0$ et $\lim_{x \rightarrow +\infty} \frac{\sqrt{1+\frac{1}{x}}+1}{\sqrt{1+\frac{1}{x^2}}+1+1} = 2$. Donc $\lim_{x \rightarrow +\infty} \frac{\sqrt{x^2+1}-x}{\sqrt{x+1}-\sqrt{x}} = 0$.

C' est plus simple en 2^{ème} année avec de nouvelles méthodes.